FEM勉強会（第４回）
その２、はり、平板、シェル解析について

平成22年11月10日

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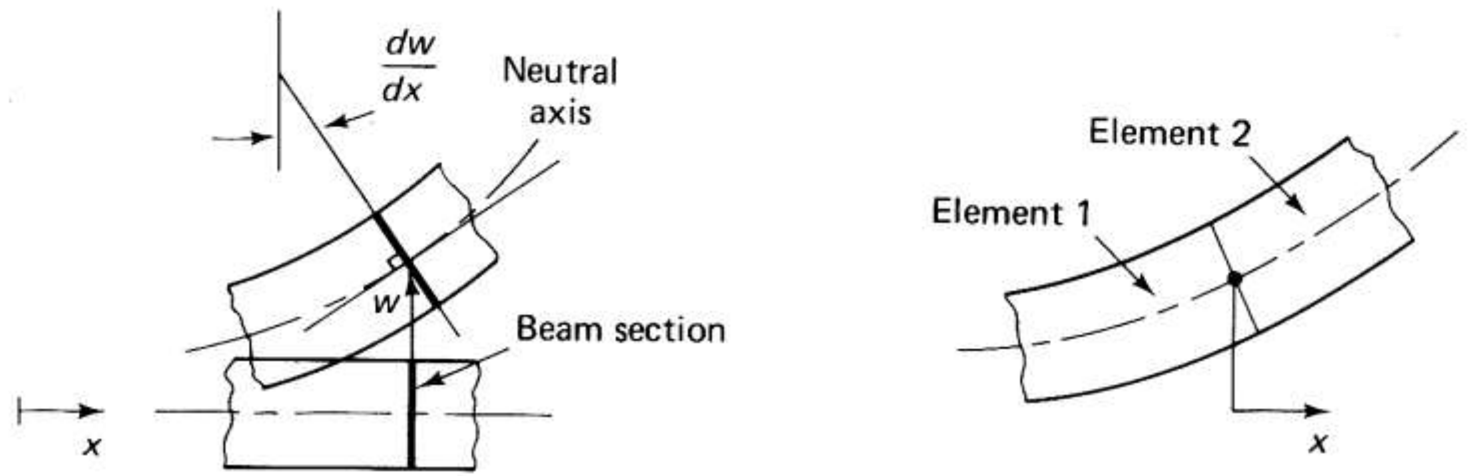
はりの理論について

Bernoulli-Eulerはり理論



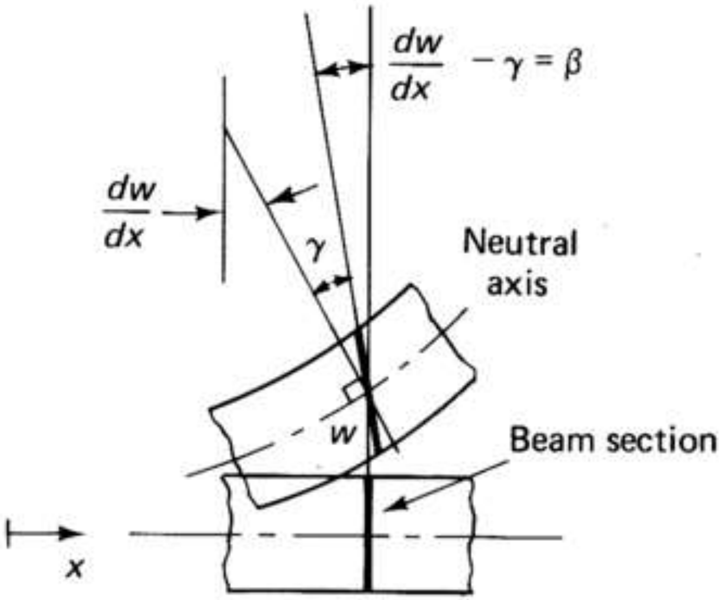
Deformation of cross-section

Boundary conditions between beam elements

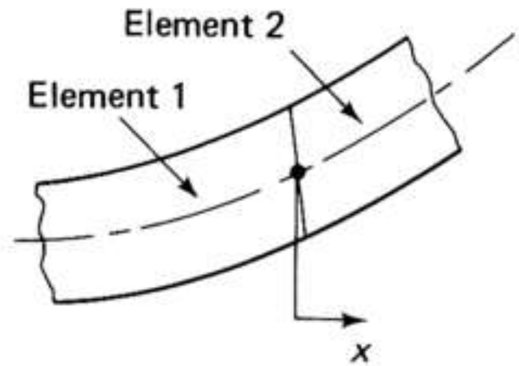
$$w\Big|_{x^{-0}} = w\Big|_{x^{+0}}\ ;\quad \frac{dw}{dx}\Big|_{x^{-0}} = \frac{dw}{dx}\Big|_{x^{+0}}$$

(a) Beam deformations excluding shear effect

Timoshenkoはり理論



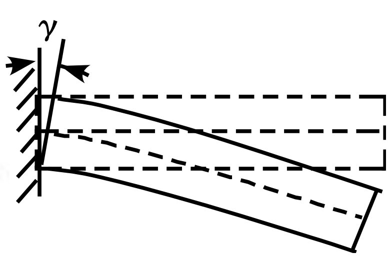
Deformation of cross-section



Boundary conditions between beam elements

$$w\Big|_{x-0} = w\Big|_{x+0}$$

$$\beta\Big|_{x-0} = \beta\Big|_{x+0}$$



Bernoulli-Eulerはり理論

w： たわみ、 たわみ角： $\theta = \dfrac{dw}{dx}$ 曲率： $\phi = \dfrac{d\theta}{dx} = \dfrac{d^2 w}{dx^2}$

弾性則： $M = EI\phi$ $Q = \dfrac{dM}{dx}$

Timoshenkoはり理論

せん断変形： γ 回転角： $\beta = \dfrac{dw}{dx} - \gamma$ 曲率： $\phi = \dfrac{d\beta}{dx}$,

弾性則： $\tau = \dfrac{Q}{\kappa A}$ $\kappa = 5/6$ $\gamma = \dfrac{\tau}{G}$

$$M = EI\phi = EI\frac{d\beta}{dx} = EI\left(\frac{d^2 w}{dx^2} - \frac{d\gamma}{dx}\right)$$

ひずみエネルギー

$$W_i = \frac{EI}{2}\int_0^L \phi^2 dx + \frac{GA\kappa}{2}\int_0^L \gamma^2 dx$$

ポテンシャルエネルギーと

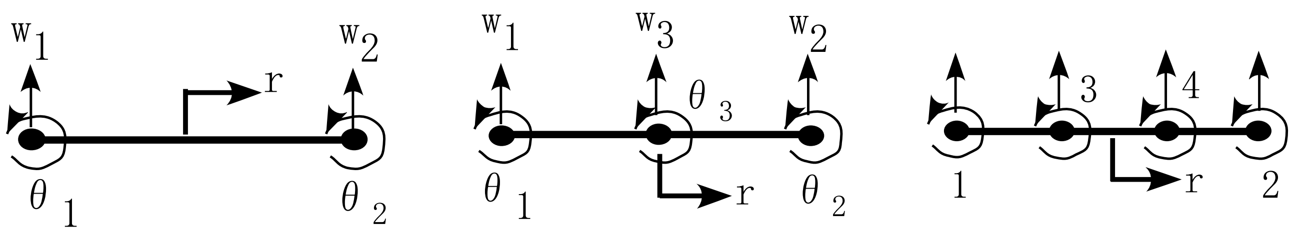
$$\pi = \frac{EI}{2}\int_0^L \left(\frac{d\beta}{dx}\right)^2 dx + \frac{GA\kappa}{2}\int_0^L \left(\frac{dw}{dx} - \beta\right)^2 dx - \int_0^L pwdx$$

極小の原理

$$EI\int_0^L \left(\frac{d\beta}{dx}\right)\cdot\delta\left(\frac{d\beta}{dx}\right)dx + GA\kappa\int_0^L \left(\frac{dw}{dx} - \beta\right)\cdot\delta\left(\frac{dw}{dx} - \beta\right)dx - \int_0^L p\cdot\delta w dx = 0$$

補間関数

$$w = \sum_{i=1}^{q} h_i w_i \qquad \beta = \sum_{i=1}^{q} h_i \theta_i$$



(a) q=2 (b) q=3 (c) q=4

FEMの定式化

$$w = \mathbf{H}_w \hat{\mathbf{u}} \qquad \beta = \mathbf{H}_\beta \hat{\mathbf{u}}$$

$$\hat{\mathbf{u}} = \left[w_1 \ldots\ldots\ldots w_q \quad \theta_1 \ldots\ldots\ldots \theta_q \right]$$

$$\mathbf{H}_w = \left[h_1 \ldots\ldots h_q \quad 0 \ldots\ldots\ldots 0 \right]^T \qquad \mathbf{H}_\beta = \left[0 \ldots\ldots 0 \quad h_1 \ldots\ldots\ldots h_q \right]^T$$

$$\frac{\partial w}{\partial x} = \mathbf{B}_w \hat{\mathbf{u}} \qquad \frac{\partial \beta}{\partial x} = \mathbf{B}_\beta \hat{\mathbf{u}}$$

$$\mathbf{B}_w = \mathbf{J}^{-1} \left[\frac{\partial h_1}{\partial r} \ldots\ldots\ldots\ldots \frac{\partial h_q}{\partial r} \; 0 \ldots\ldots 0 \right] \qquad \mathbf{B}_\beta = \mathbf{J}^{-1} \left[0 \ldots\ldots 0 \; \frac{\partial h_1}{\partial r} \ldots\ldots\ldots\ldots \frac{\partial h_q}{\partial r} \right]$$

$$\mathbf{K} = EI \int_{-1}^{1} \mathbf{B}_\beta^T \mathbf{B}_\beta \det \mathbf{J} + GA\kappa \int_{-1}^{1} (\mathbf{B}_w - \mathbf{H}_\beta)^T (\mathbf{B}_w - \mathbf{H}_\beta) \det \mathbf{J} dr$$

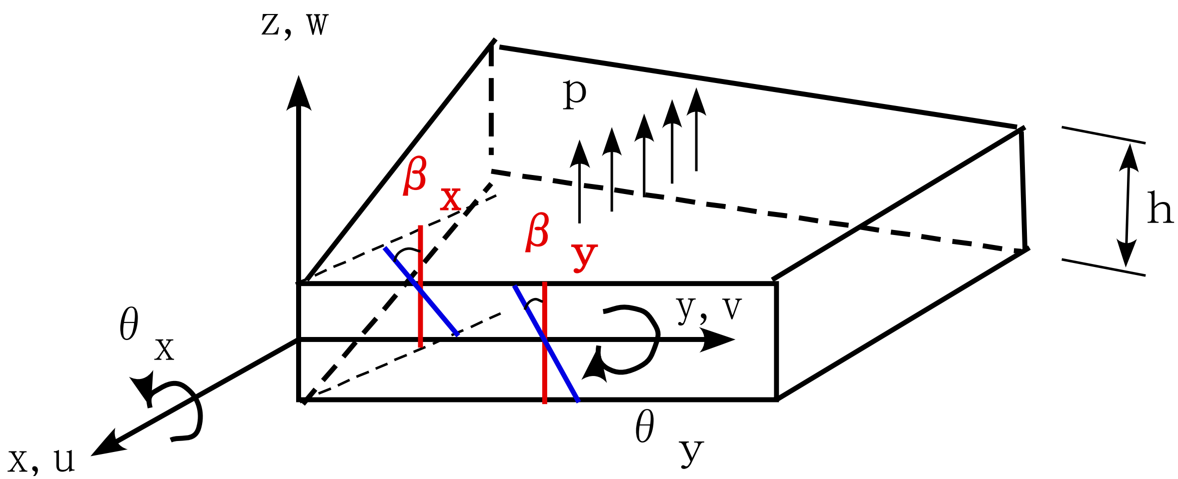
$$\mathbf{R} = \int_{-1}^{1} \mathbf{H}_w^T p \det \mathbf{J} dr$$

剛性方程式： $K\hat{u} = R$

平板のFEM解析

$$u = z\beta_x(x, y) \qquad v = -z\beta_y(x, y) \qquad w = w(x, y)$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = z\begin{bmatrix} \partial\beta_x / \partial x \\ -\partial\beta_y / \partial y \\ \partial\beta_x / \partial y - \partial\beta_y / \partial x \end{bmatrix} \qquad \begin{bmatrix} \gamma_{yz} \\ \gamma_{xz} \end{bmatrix} = \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix}$$



$\beta_x \quad \beta_y$ 回転角

FEMの定式化

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = z \cdot \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \cdot \begin{bmatrix} \partial\beta_x / \partial x \\ -\partial\beta_y / \partial y \\ \partial\beta_x / \partial y - \partial\beta_y / \partial x \end{bmatrix}$$

$$\begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} = \frac{E}{2(1+\nu)} \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix}$$

$$\boldsymbol{\varphi} = \begin{bmatrix} \partial\beta_x / \partial x \\ -\partial\beta_y / \partial y \\ \partial\beta_x / \partial y - \partial\beta_y / \partial x \end{bmatrix}$$

$$\pi = \frac{1}{2} \int_{A_0} \int_{-h/2}^{h/2} \begin{bmatrix} \varepsilon_x & \varepsilon_y & \gamma_{xy} \end{bmatrix} \cdot \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \cdot dz dA_0 + \frac{\kappa}{2} \int_{A_0} \int_{-h/2}^{h/2} \begin{bmatrix} \gamma_{yz} & \gamma_{xz} \end{bmatrix} \cdot \begin{bmatrix} \tau_{yz} \\ \tau_{xz} \end{bmatrix} \cdot dz dA_0 - \int_{A_0} wp dA_0$$

$$\pi = \frac{1}{2} \int_{A_0} \boldsymbol{\varphi}^T \mathbf{C}_b \boldsymbol{\varphi} dA_0 + \frac{1}{2} \int_{A_0} \boldsymbol{\gamma}^T \mathbf{C}_s \boldsymbol{\gamma} dA_0 - \int_{A_0} wp dA_0$$

$$\boldsymbol{\gamma} = \begin{bmatrix} \partial w / \partial y - \beta_y \\ \partial w / \partial x + \beta_x \end{bmatrix}$$

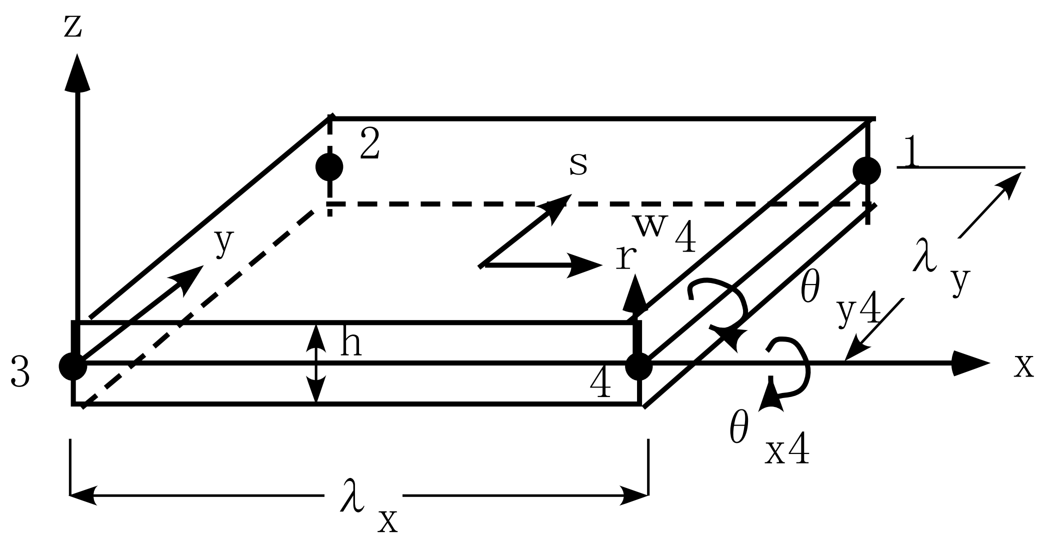
$\hat{\mathbf{u}} = [w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \ldots\ldots\ldots \theta_{y4}]^T$

ポテンシャルエネルギー極小の条件

$$\int_{A_0} \delta\boldsymbol{\varphi}^T \mathbf{C}_b \boldsymbol{\varphi} dA_0 + \int_{A_0} \delta\boldsymbol{\gamma}^T \mathbf{C}_s \boldsymbol{\gamma} dA_0 - \int_{A_0} \delta w p dA_0 = 0$$

$$\boldsymbol{\varphi}(r,s) = \mathbf{B}_\phi \hat{\mathbf{u}} \qquad \boldsymbol{\gamma}(r,s) = \mathbf{B}_s \hat{\mathbf{u}} \qquad w(r,s) = \mathbf{H}_w \hat{\mathbf{u}}$$

$$\hat{\mathbf{u}} = [w_1 \quad \theta_{x1} \quad \theta_{y1} \quad w_2 \ldots\ldots\ldots \theta_{y4}]^T$$



$$h_1 = \frac{1}{4}(1+r)(1+s) \quad h_2 = \frac{1}{4}(1-r)(1+s) \quad h_3 = \frac{1}{4}(1-r)(1-s) \quad h_4 = \frac{1}{4}(1+r)(1-s)$$

剛性方程式 $\mathbf{K}\hat{\mathbf{u}} = \mathbf{R}_\mathbf{s}$

$$\mathbf{K} = \int_{-1}^{1}\int_{-1}^{1} (\mathbf{B}_\phi^T \mathbf{C}_b \mathbf{B}_\phi + \mathbf{B}_s^T \mathbf{C}_s \mathbf{B}_s) \det \mathbf{J} \cdot drds$$

$$\mathbf{R}_s = \int_{-1}^{1}\int_{-1}^{1} \mathbf{H}_w^T p \det \mathbf{J} \cdot drds$$

境界条件:

単純支持辺: $w = 0$

固定辺: $w = 0$ $\theta_x = \beta_x = 0, \theta_y = \beta_y = 0$

自由辺: なし

シェルの

FEM解析

要素：板曲げ＋シャイベ

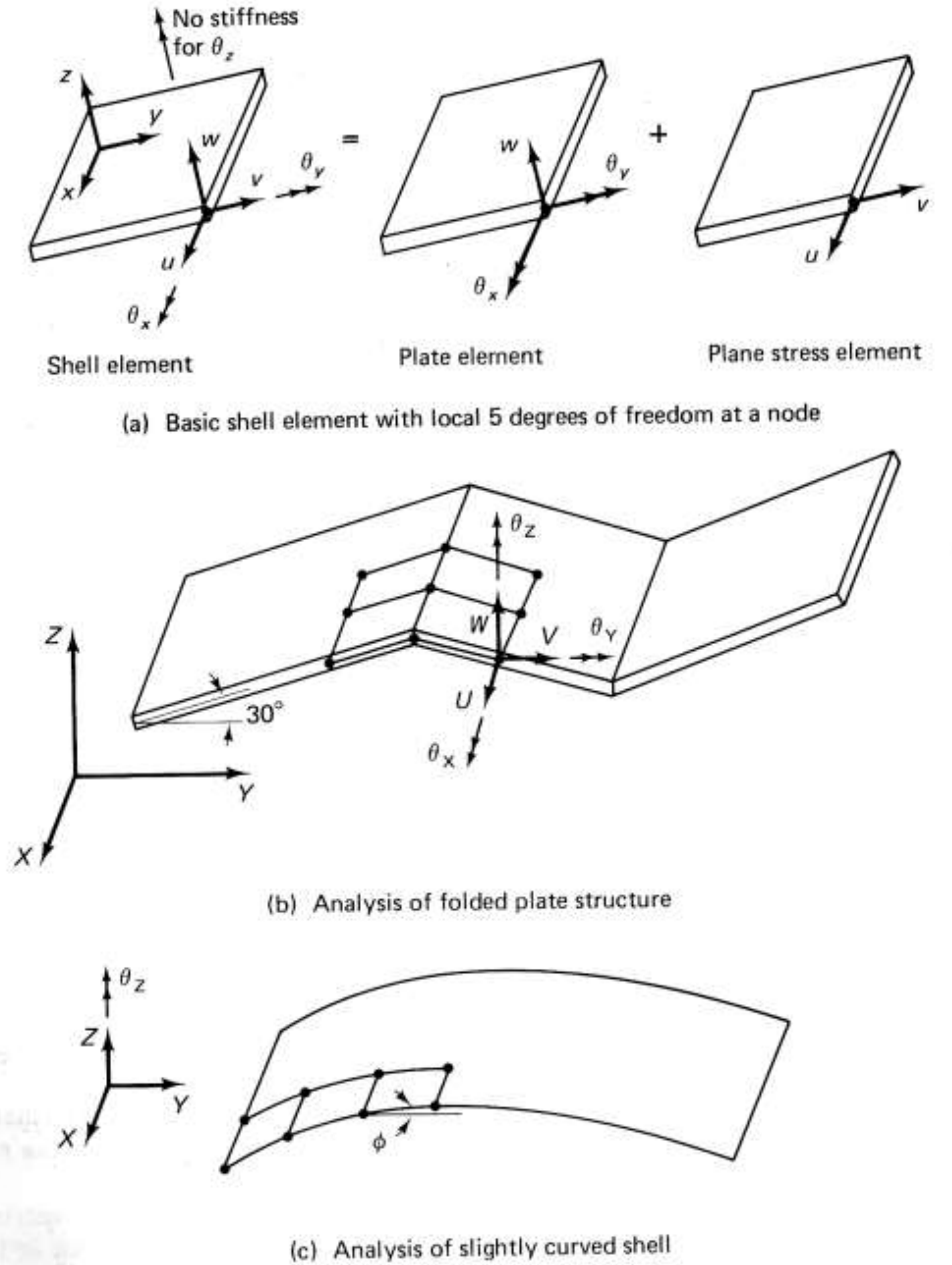


FIGURE 4.19 Use of a flat shell element.

四辺形要素の場合

$$\underset{20\times20}{\hat{\mathbf{K}}_{shell}} = \begin{bmatrix} \mathbf{K}_B & \mathbf{0} \\ \mathbf{0} & \mathbf{K}_M \end{bmatrix}$$

20x20

$$\mathbf{K}_{shell} = \mathbf{T}^T \hat{\mathbf{K}}_{shell} \mathbf{T}$$

24x24

$$\mathbf{K}_{shell} = \begin{bmatrix} \hat{\mathbf{K}}_{shell} & \mathbf{0} \\ \mathbf{0} & k\mathbf{I} \end{bmatrix}$$

24x24