

**FEM勉強会（第4回）**  
**その3、材料非線形問題の**  
**FEM解析について**

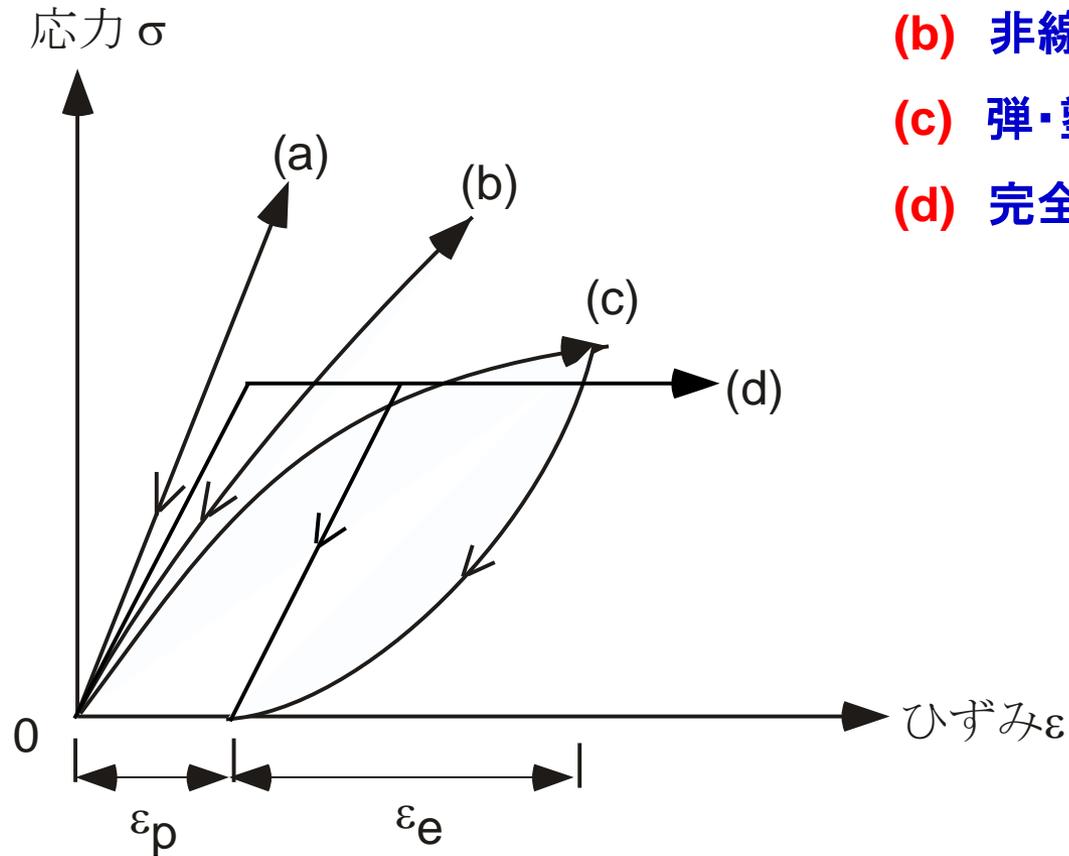
**平成22年11月10日**

**園田 恵一郎**



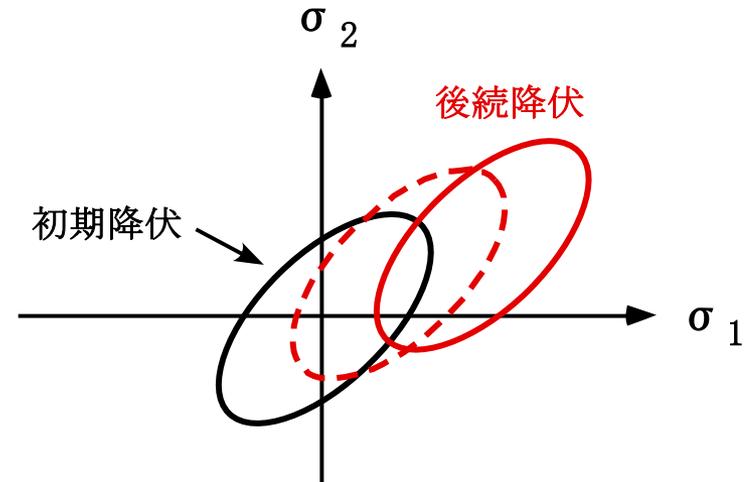
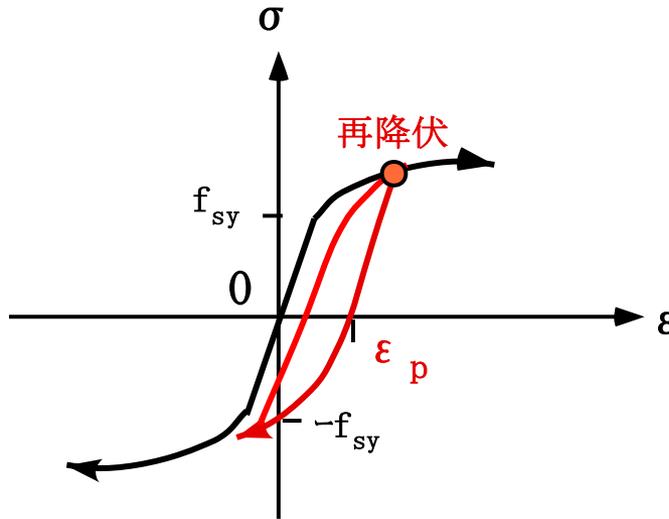
# 材料非線形問題とは？

- (a) 線形弾性
- (b) 非線形弾性
- (c) 弾・塑性
- (d) 完全塑性

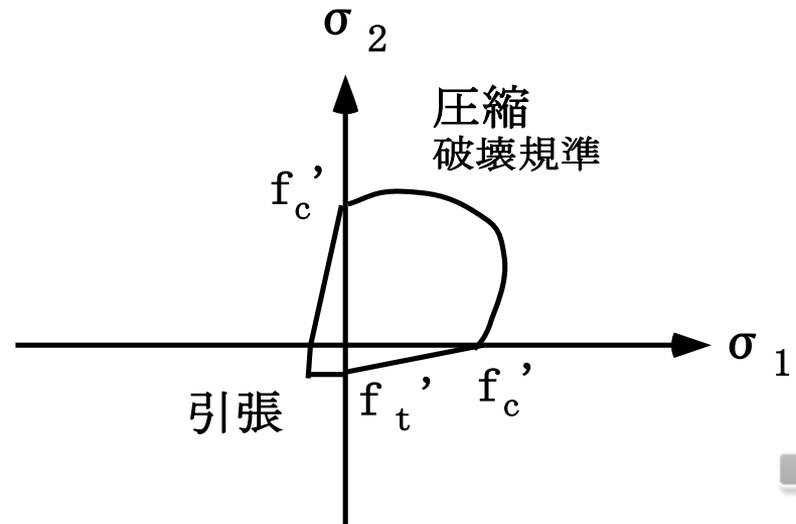
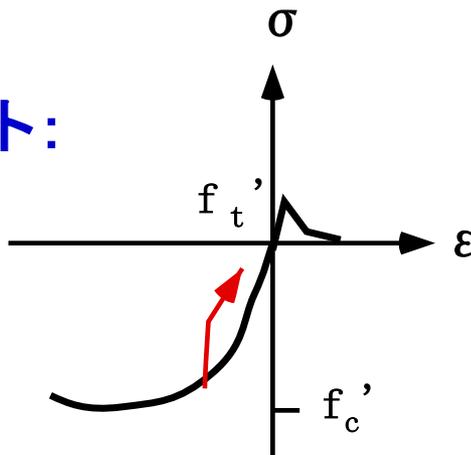


# モデルの選択: 非線形弾性モデルか、弾塑性モデルか?

鋼:



コンクリート:



# 非線形弾性モデルに対するFEM解析

**t** 時刻のひずみと応力

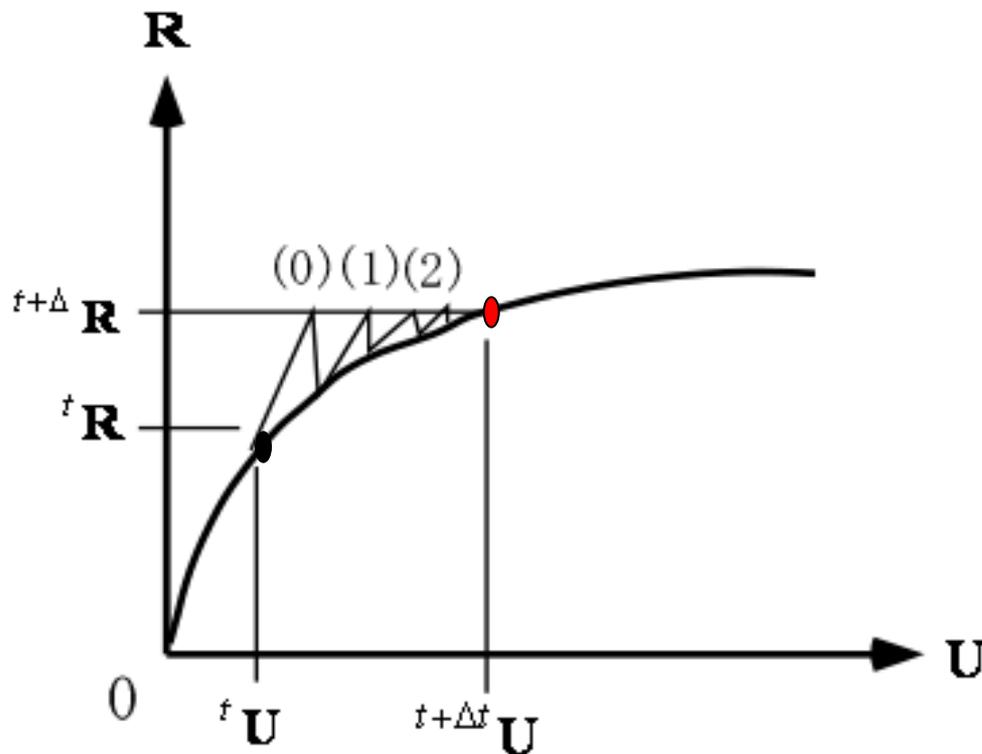
$${}^t \boldsymbol{\varepsilon}^{(a)} = {}^t \mathbf{B}^{(a)} {}^t \mathbf{U}$$

$${}^t \boldsymbol{\sigma}^{(a)} = {}^t \mathbf{C} \cdot {}^t \boldsymbol{\varepsilon}^{(a)}$$

**剛性方程式 (外力と内力のつりあい)**

$${}^t \mathbf{R} = {}^t \mathbf{F}$$

$${}^t \mathbf{R} = {}^t \mathbf{K} {}^t \mathbf{U}$$



$${}^t \mathbf{F} = \sum_{(a)} \int_{V^{(a)}} {}^t \mathbf{B}^{(a)T} {}^t \mathbf{C} {}^t \mathbf{B}^{(a)T} dV^{(a)} \cdot {}^t \mathbf{U} = \sum_{(a)} \int_{V^{(a)}} {}^t \mathbf{B}^{(a)T} {}^t \boldsymbol{\sigma} dV^{(a)}$$

$${}^t \mathbf{K} = \sum_{(a)} \int_{V^{(a)}} \mathbf{B}^{(a)T} {}^t \mathbf{C} \mathbf{B} dV^{(a)}$$



# 修正ニュートン法による反復・収束計算

$${}^{t+\Delta t} \mathbf{R} - {}^{t+\Delta t} \mathbf{F} = \mathbf{0} \quad {}^{t+\Delta t} \mathbf{F} = \sum_{(a)} \int_{V^{(a)}} \mathbf{B}^T {}^{t+\Delta t} \boldsymbol{\sigma} dV^{(a)}$$

初期条件:  $i=0$

$${}^{t+\Delta t} \mathbf{U}^{(0)} = {}^t \mathbf{U} \quad ; \quad {}^{t+\Delta t} \mathbf{F}^{(0)} = {}^t \mathbf{F}$$

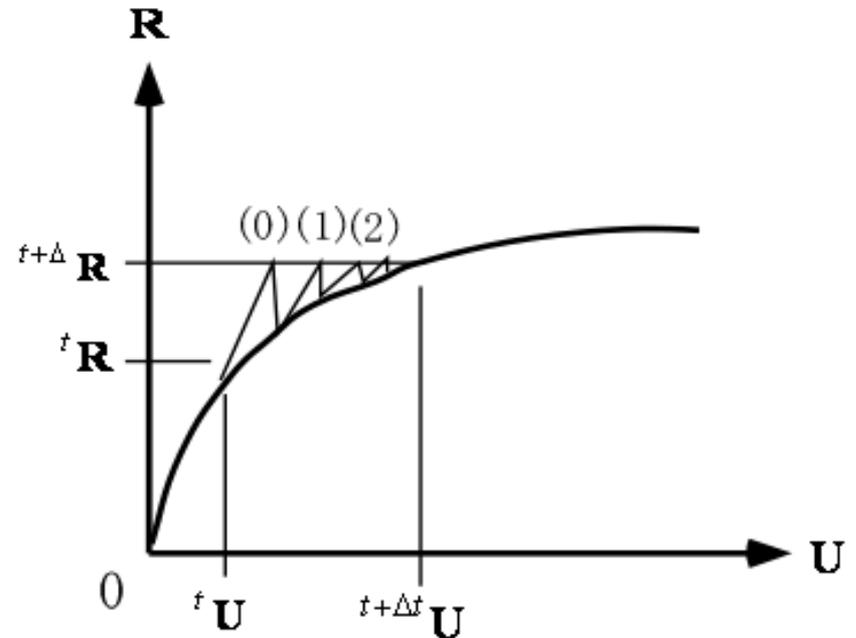
$${}^t \mathbf{K} \Delta \mathbf{U}^{(i)} = {}^{t+\Delta t} \mathbf{R} - {}^{t+\Delta t} \mathbf{F}^{(i-1)}$$

$${}^{t+\Delta t} \mathbf{U}^{(i)} = {}^{t+\Delta t} \mathbf{U}^{(i-1)} + \Delta \mathbf{U}^{(i)}$$

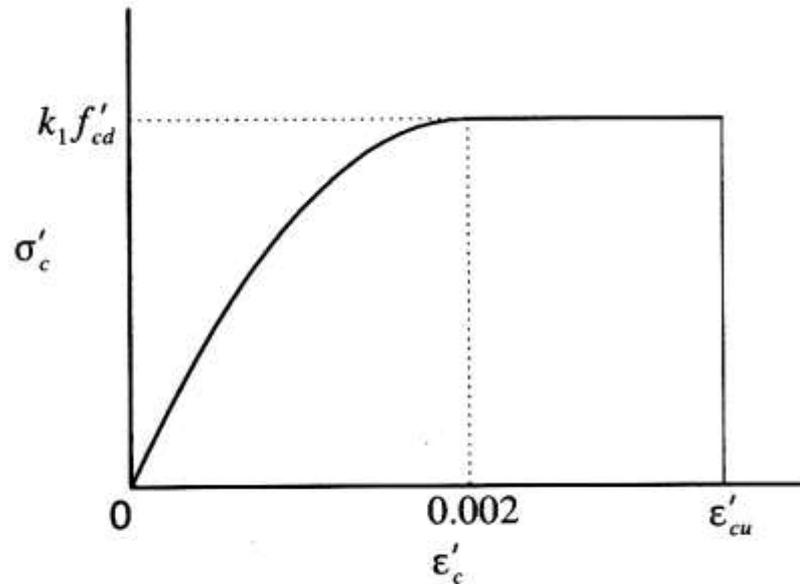
$${}^{t+\Delta t} \mathbf{F}^{(i)} = \sum_{(a)} \int_{V^{(a)}} \mathbf{B}^T {}^{t+\Delta t} \boldsymbol{\sigma}^{(i)} dV^{(a)}$$

$${}^{t+\Delta t} \boldsymbol{\sigma}^{(i)} = {}^{t+\Delta t} \boldsymbol{\sigma}^{(i-1)} + {}^t \mathbf{C}^{(i)} \mathbf{B} \Delta \mathbf{U}^{(i)}$$

$i=1,2,3,\dots$



# 1軸応力-ひずみ曲線(コンクリート標準示方書による)



$$k_1 = 1 - 0.003 f'_{ck} \quad \leq 0.85$$

$$\epsilon'_{cu} = \frac{155 - f'_{ck}}{30000} \quad 0.0025 \leq \epsilon'_{cu} \leq 0.0035$$

ここで、 $f'_{ck}$  の単位は  $\text{N}/\text{mm}^2$

曲線部の応力ひずみ式

$$\sigma'_c = k_1 f'_{cd} \times \frac{\epsilon'_c}{0.002} \times \left( 2 - \frac{\epsilon'_c}{0.002} \right)$$

図 3. 2. 1 コンクリートの応力-ひずみ曲線



# 圧縮領域のコンクリートに対する非線形弾性モデル

八面体直応力 $\sigma_0$ とせん断応力 $\tau_0$ と対応するひずみの関係

$$\sigma_0 = K_0 \varepsilon_0 (1 + a \varepsilon_0 + b \varepsilon_0^2 + c \varepsilon_0^3 + \dots)$$

$$\tau_0 = G_0 \gamma_0 (1 + a' \gamma_0 + b' \gamma_0^2 + c' \gamma_0^3 + \dots)$$

$$\sigma_0 = \frac{1}{3}(\sigma_1 + \sigma_2 + \sigma_3) \quad \tau_0 = \frac{1}{3}\sqrt{(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2}$$

$$\varepsilon_0 = \frac{1}{3}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \quad \gamma_0 = \frac{1}{3}\sqrt{(\varepsilon_1 - \varepsilon_2)^2 + (\varepsilon_2 - \varepsilon_3)^2 + (\varepsilon_3 - \varepsilon_1)^2}$$

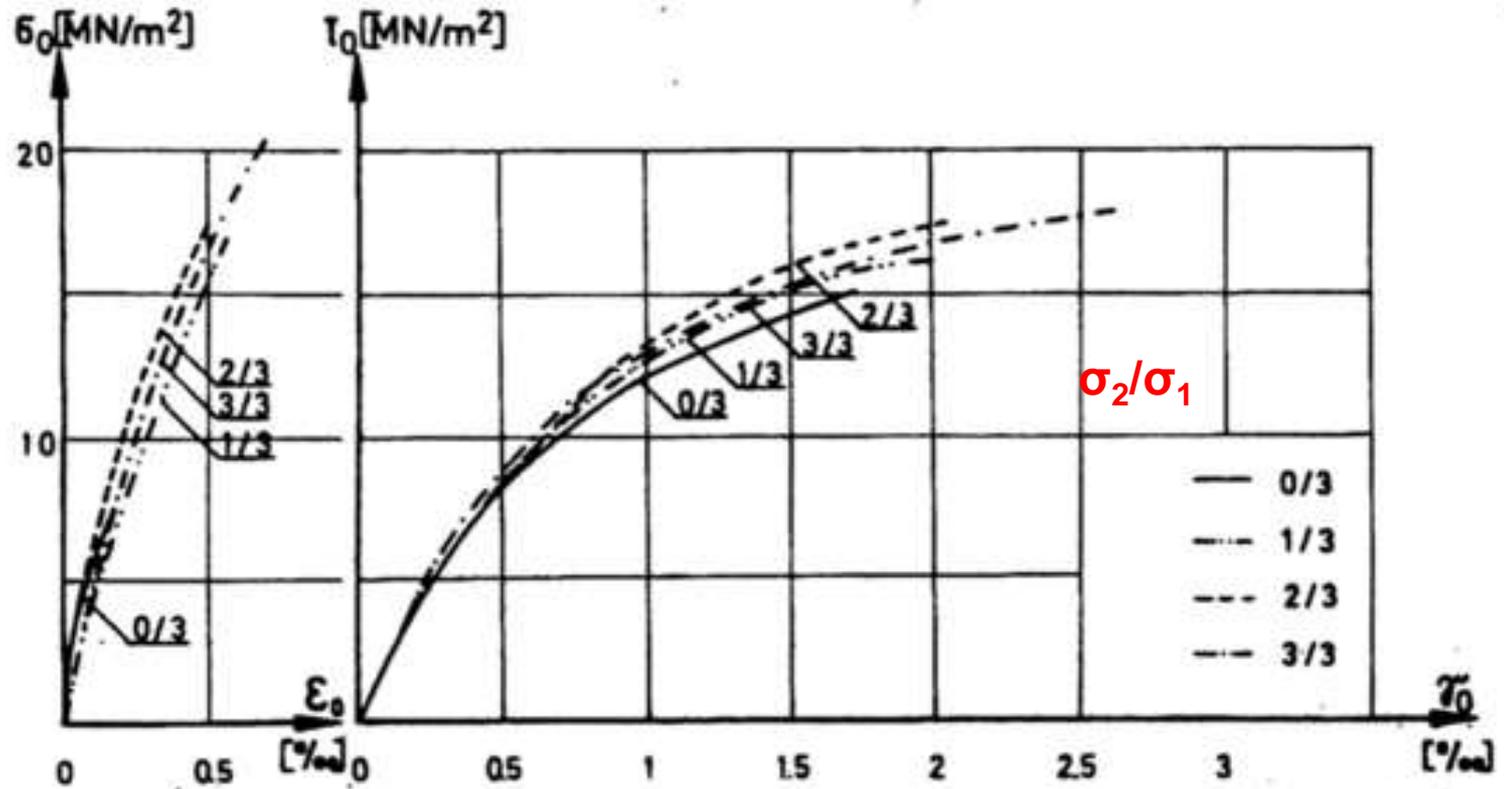
応力増分とひずみの関係

$$d\sigma_0 = K_0 d\varepsilon_0 (1 + 2a\varepsilon_0 + 3b\varepsilon_0^2 + 4c\varepsilon_0^3 + \dots)$$

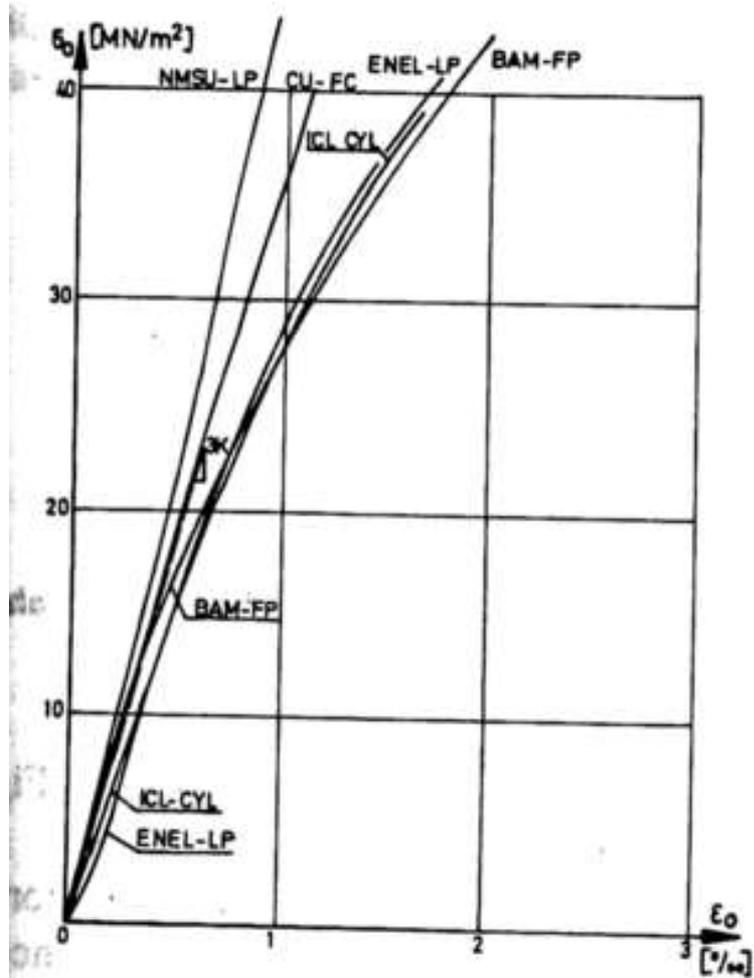
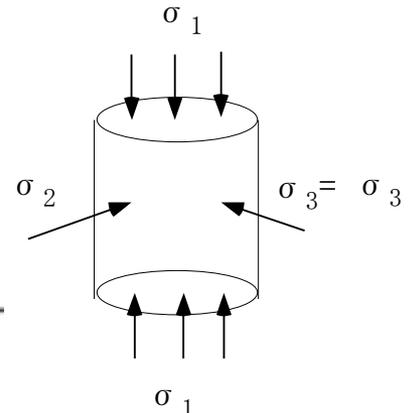
$$d\tau_0 = G_0 d\gamma_0 (1 + 2a'\gamma_0 + 3b'\gamma_0^2 + 4c'\gamma_0^3 + \dots)$$



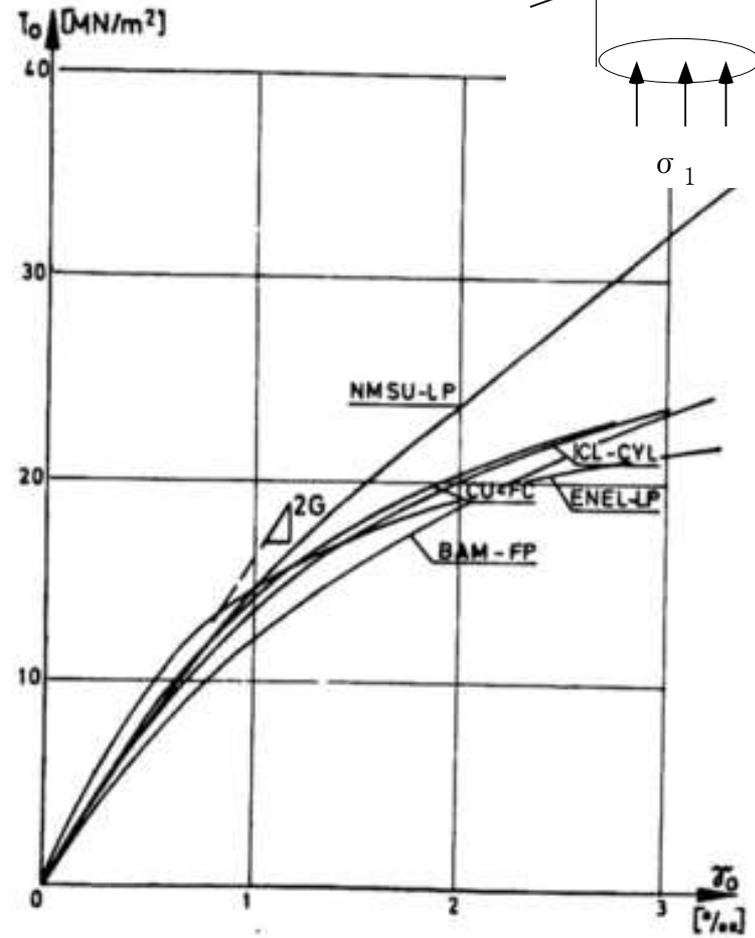
## 2軸圧縮応力状態でのコンクリート



# 3軸圧縮応力状態でのコンクリート



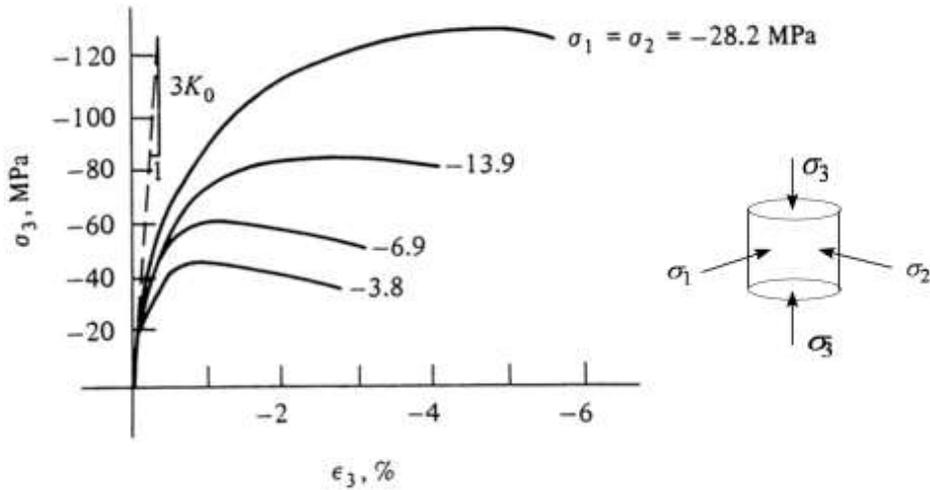
(a)



(b)



# 3軸圧縮応力状態でのコンクリートの破壊基準(実験結果)



Richard(1928)の実験データ

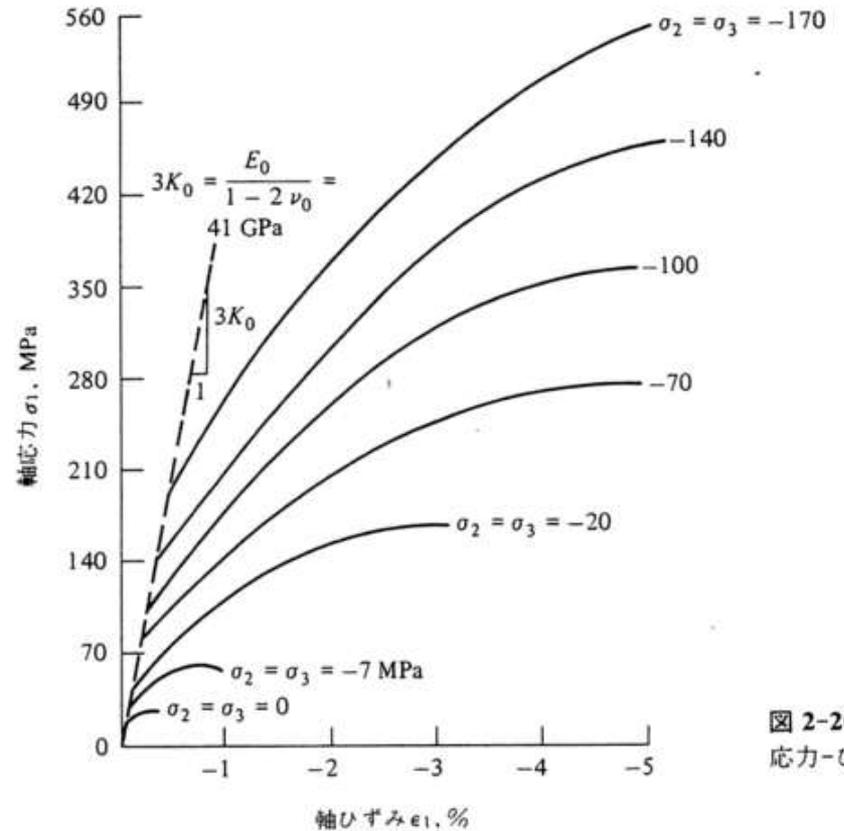
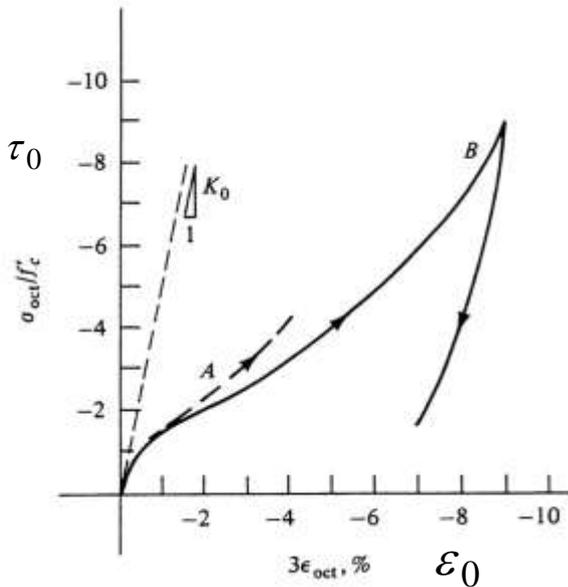
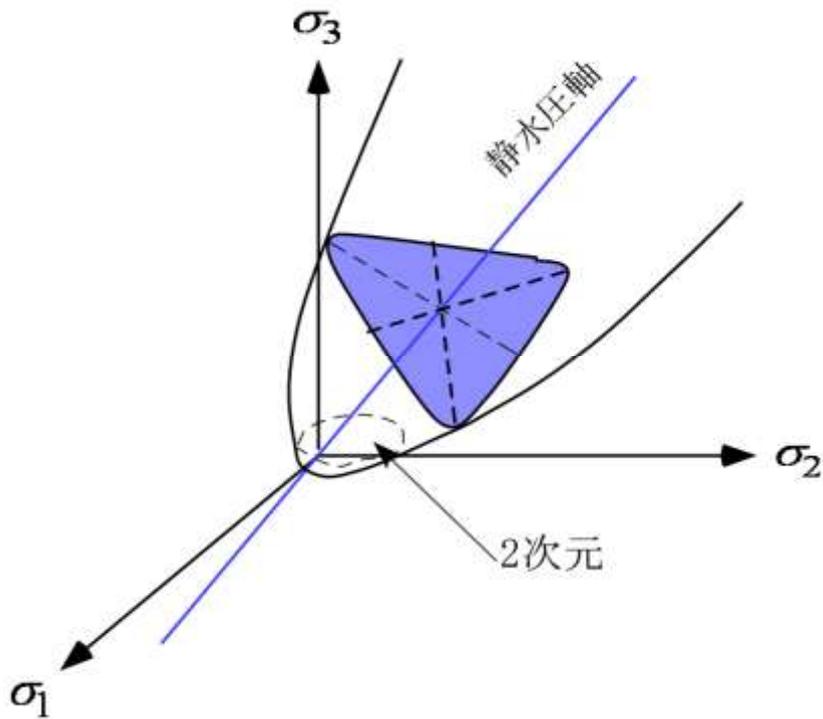


図 2-20  
応力-ひ

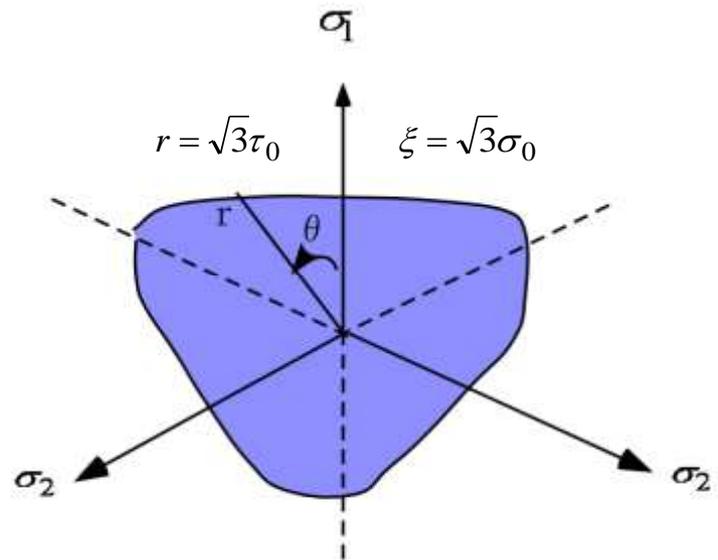
Balmer(1949)の実験データ

静水圧下でのコンクリートの挙動 , Green et.(1973)





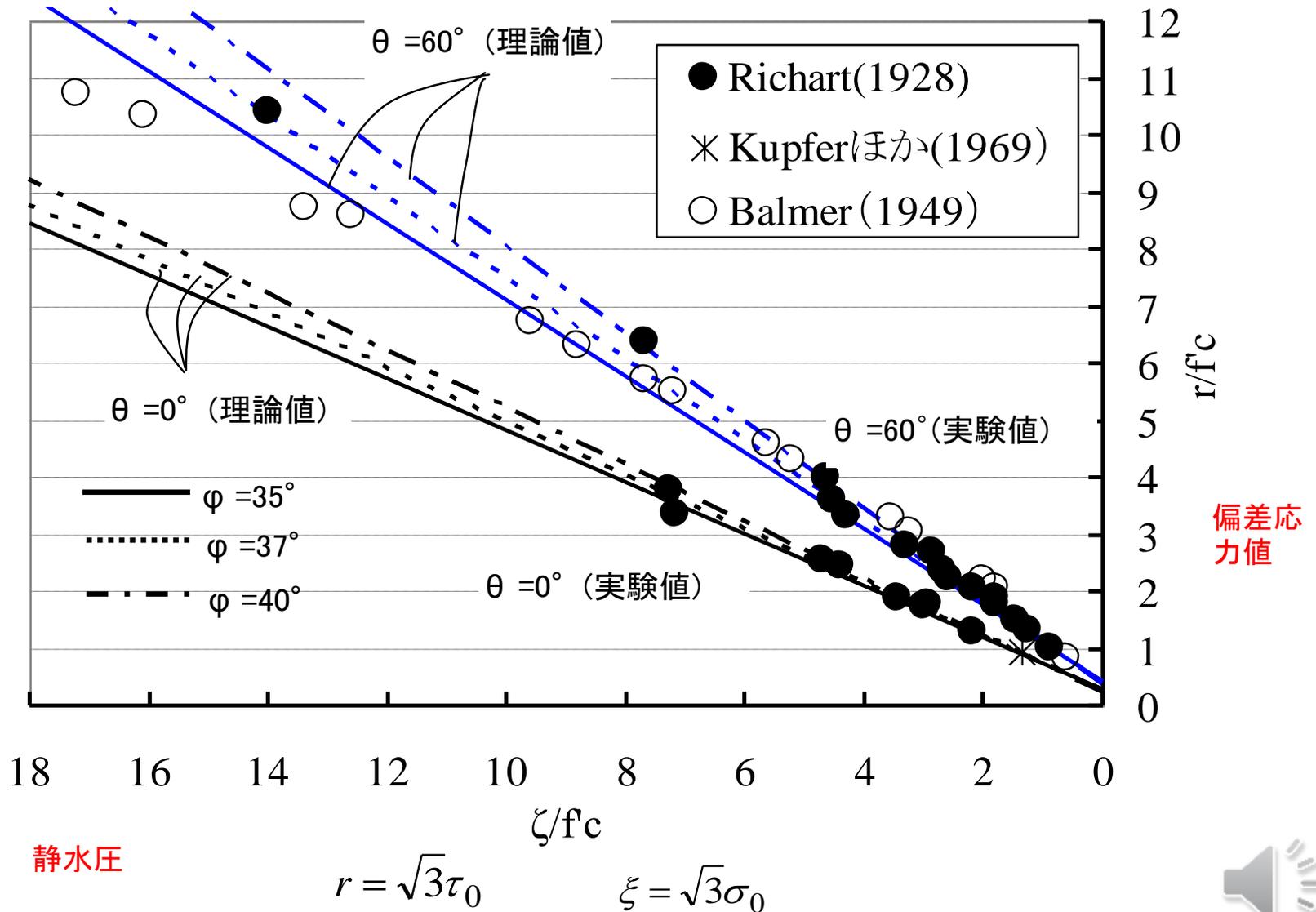
主応力空間での3軸応力状態の破壊基準



$\pi$ 平面上での破壊基準



# 3軸圧縮応力下でのコンクリートの破壊基準



# A.C.T.Chenら(1976)の提案による破壊基準の定式化(関連流動則の適用)

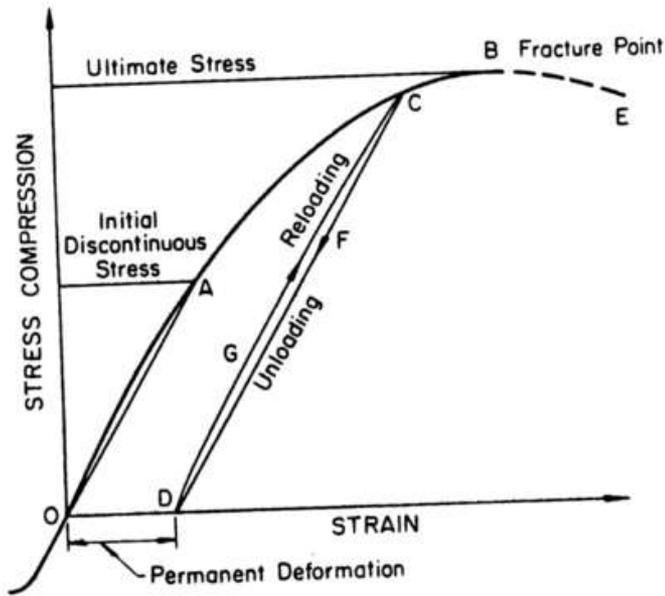


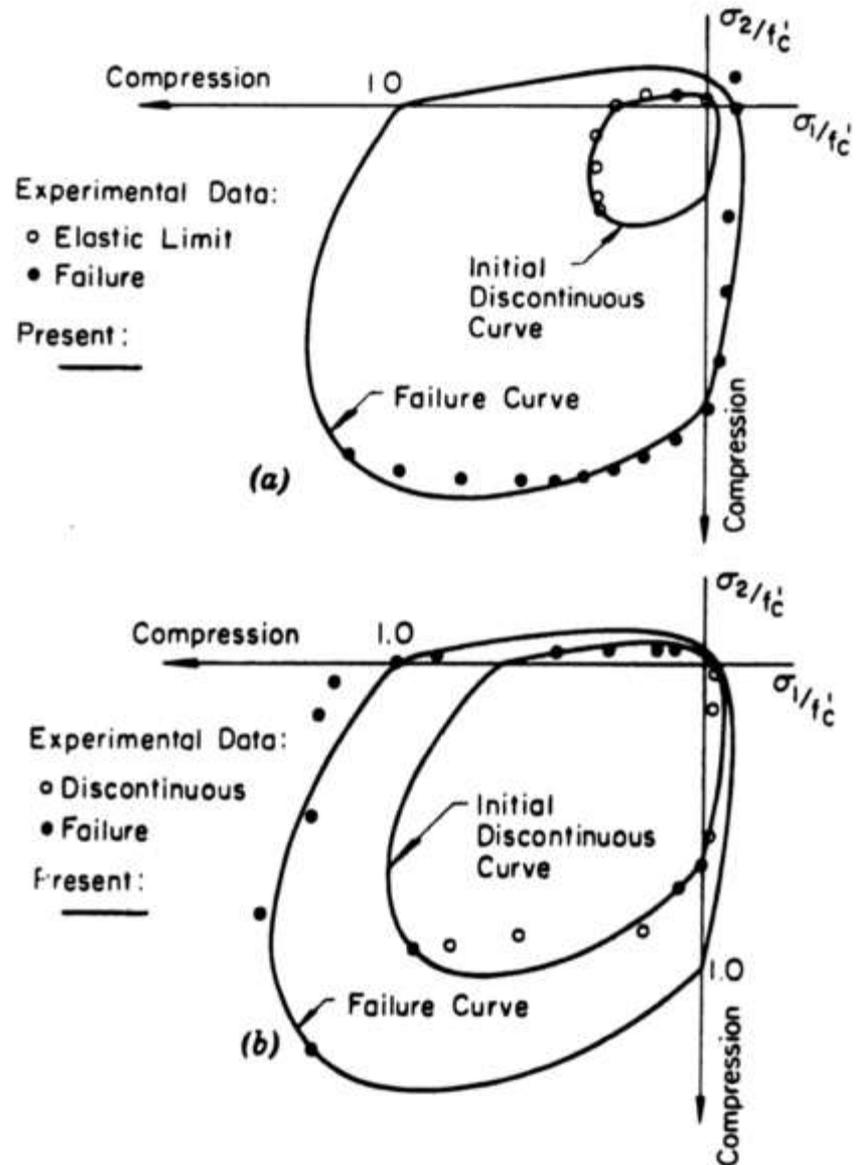
FIG. 2.—Typical Stress-Strain Curve for Concrete

Compression-Compression:

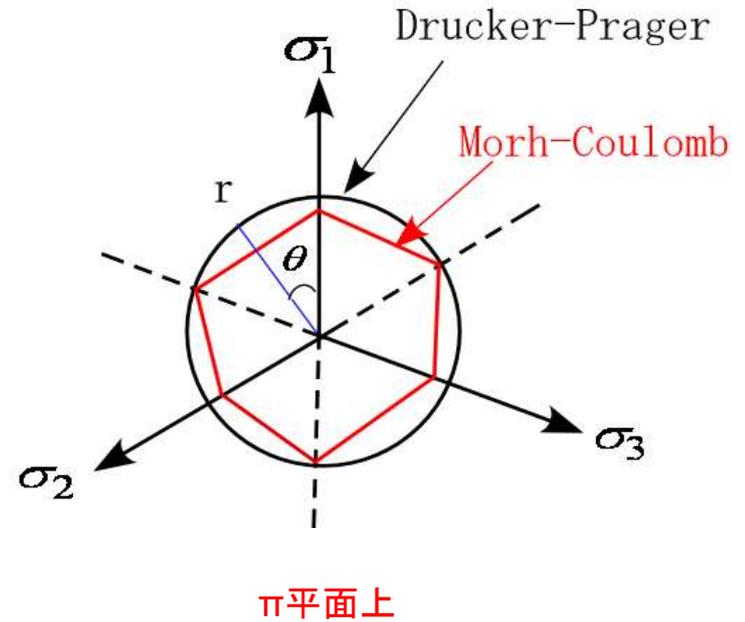
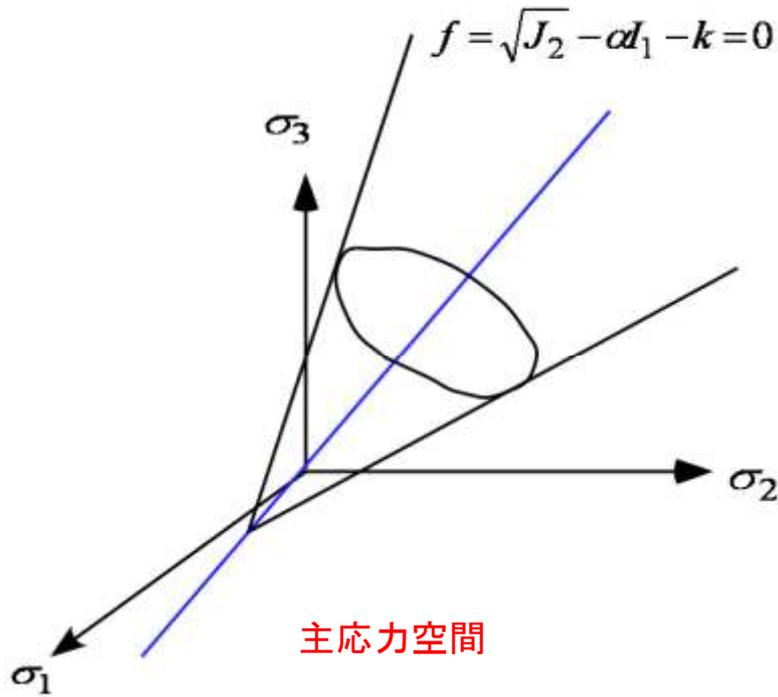
$$f_u(\sigma_{ij}) = \frac{\kappa^2}{3} J_2 - \frac{\kappa^2}{36} I_1^2 + \frac{1}{12} I_1^2 + \frac{1}{3} A_u I_1 - \tau_u^2 = 0$$

Compression-Tension:

$$f_u(\sigma_{ij}) = \frac{\kappa^2}{3} J_2 - \frac{\kappa^2}{36} I_1^2 - \frac{1}{12} I_1^2 + \frac{1}{3} A_u I_1 - \tau_u^2 = 0$$



# Drucker-Prager式



## Drucker-Prager式

$$f(I_1, J_2) = \alpha I_1 - \sqrt{J_2} - k = 0$$

## モール・クーロン式

$$f(I_1, J_2, \theta) = -\frac{1}{3} I_1 \sin \phi + \sqrt{J_2} \sin\left(\theta + \frac{\pi}{3}\right) + \sqrt{\frac{J_2}{3}} \cos\left(\theta + \frac{\pi}{3}\right) \sin \phi - c \cos \phi = 0, \quad \text{ただし } 0 \leq \theta \leq \frac{\pi}{3}$$



# Drucker-Prager式の不十分性

1軸応力状態:  $\sigma_x = \sigma, \sigma_y = \sigma_z = \tau_{xy} = \tau_{xz} = \tau_{yz} = 0$

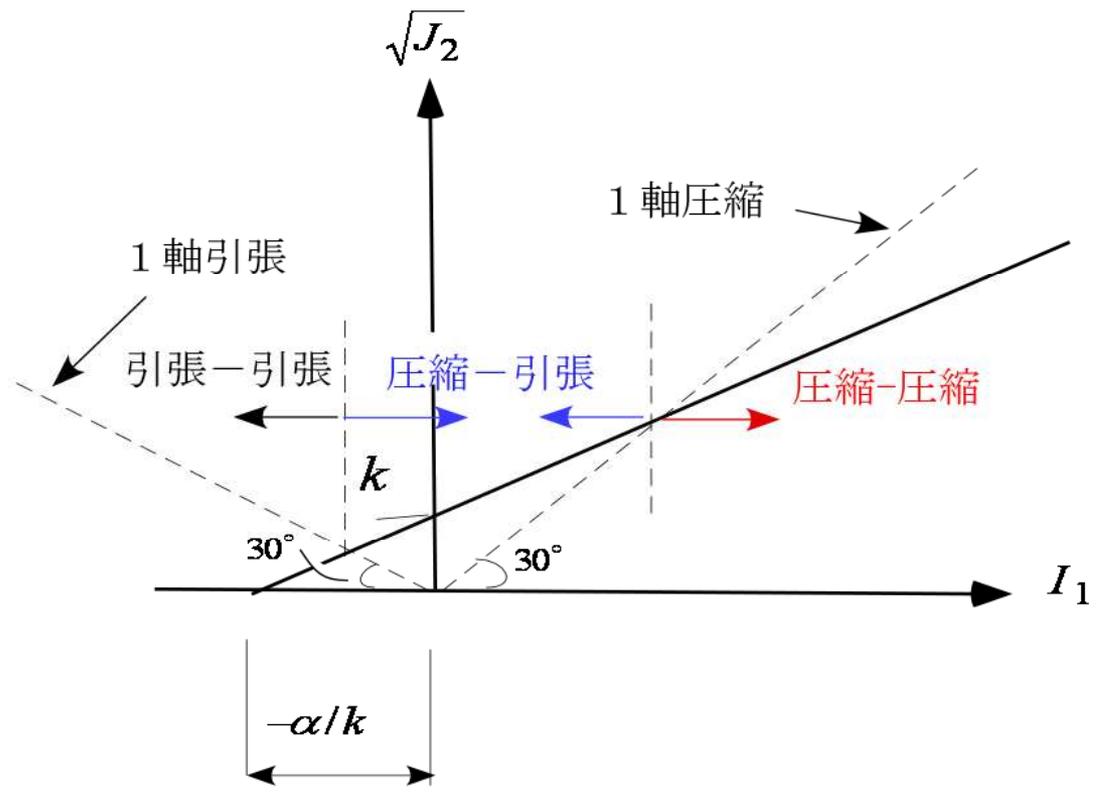
$$s_x = \frac{2}{3}\sigma, s_y = s_z = -\frac{\sigma}{3}$$

$$I_1 = \sigma, J_s = \frac{6}{2 \times 9} \sigma^2, \sqrt{J_2} = \frac{\sigma}{\sqrt{3}}$$

$$\tan\theta = \frac{\sqrt{J_2}}{I_1} = \frac{1}{\sqrt{3}}, \theta = \pi/6$$

問題点:

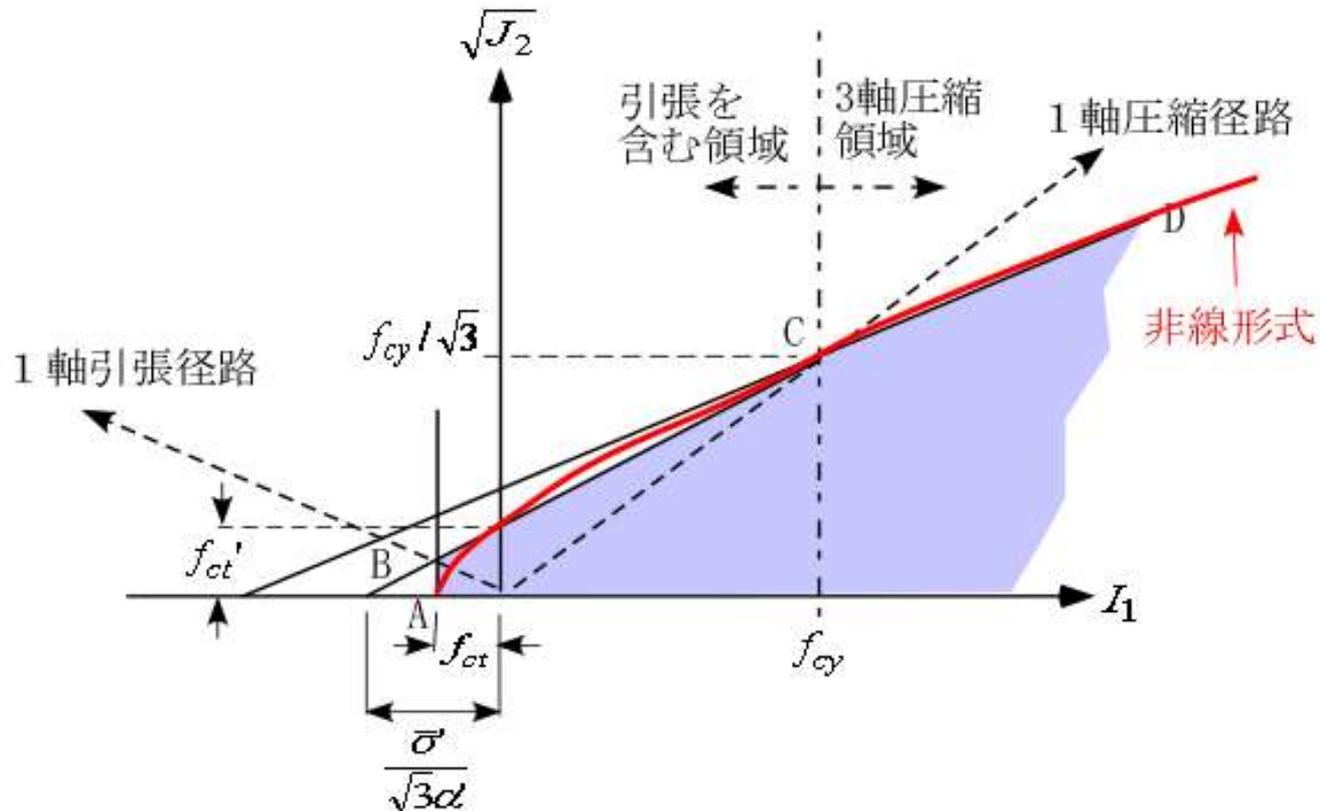
- (1) 圧縮域と引張域とが同一
- (2) 3軸引張強度が過大



# Drucker-Prager式の修正

- (1) 区分線形式, A-B: tension-cutoff, B-C, C-D
- (2) 非線形式

$$\left\{ \begin{array}{l} f_1 = \sqrt{J_2} - \alpha I_1 - \frac{\bar{\sigma}}{\sqrt{3}} = 0 \\ f_2 = \sqrt{J_2} - \alpha' I_1 - \frac{\bar{\sigma}'}{\sqrt{3}} = 0 \\ f_3 = I_1 + f_{ct} = 0 \quad \text{:Tension cutoff} \end{array} \right.$$



## 非線形式の一提案(2次放物線式)

$$f(I_1, J_2) = I_1 + a_1 \sqrt{J_2} + a_2 J_2 - k = 0$$

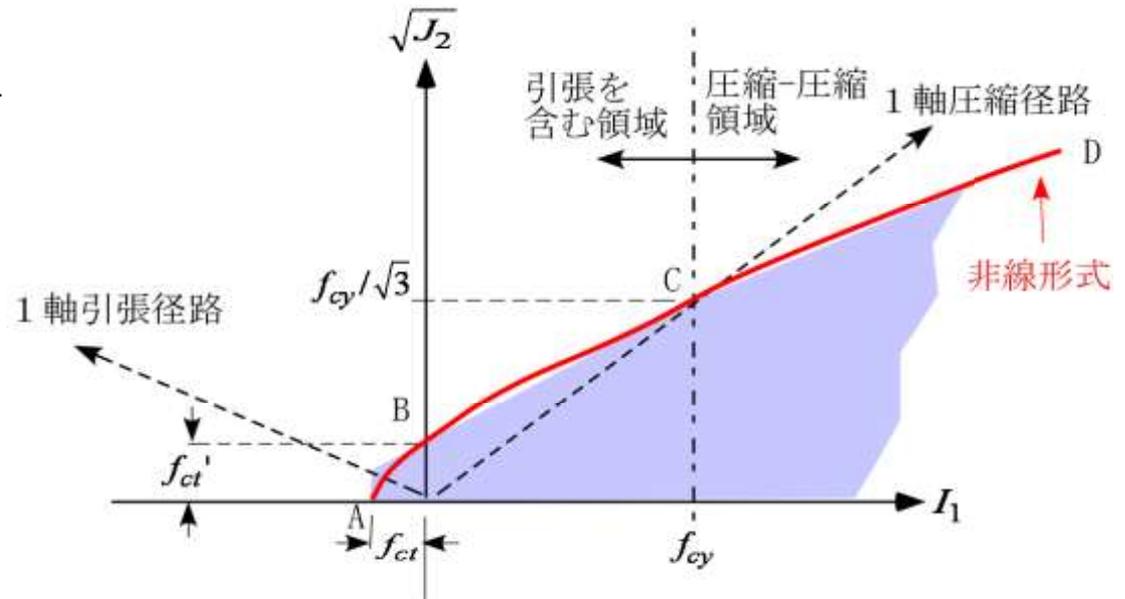
$f_{ct}$  : 3軸引張強度,  $f_{ct}'$  : 純せん断強度,  $f_{cy}$  : 1軸圧縮強度

$$\kappa = -f_{ct}$$

$$a_1 = \frac{1}{\Delta} \left\{ -\frac{1}{3} f_{ct} f_{cy}^2 + f_{ct}'^2 (f_{ct} + f_{cy}) \right\}$$

$$a_2 = \frac{1}{\Delta} \left\{ \frac{1}{\sqrt{3}} f_{ct} f_{cy} - f_{ct}' (f_{ct} + f_{cy}) \right\}$$

$$\Delta = \frac{f_{ct}'}{3\sqrt{3}} (\sqrt{3} f_{cy} - f_{ct}'^2)$$



### 等価塑性ひずみ増分

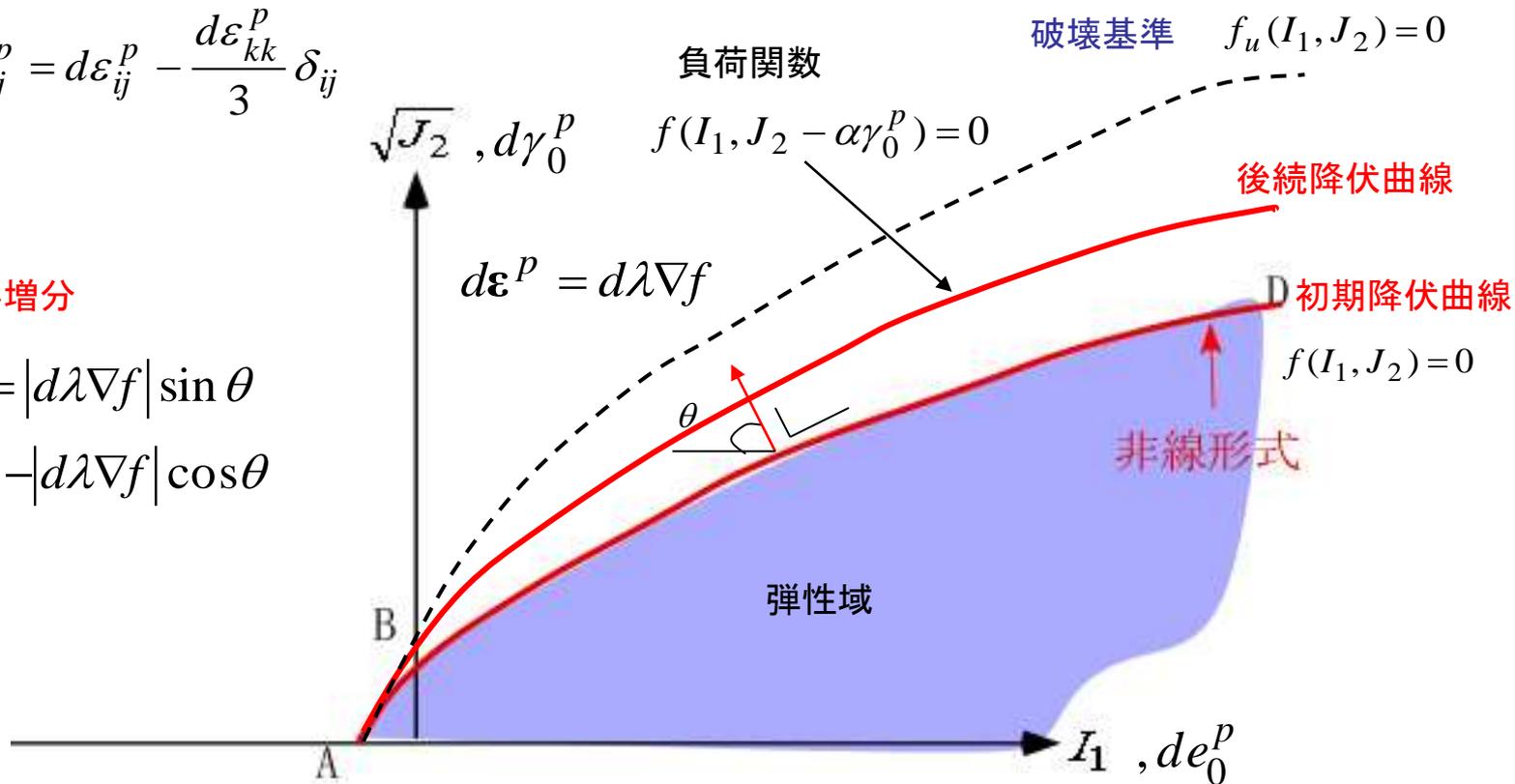
$$I_1 \rightarrow de_0^p = d\varepsilon_x^p + d\varepsilon_y^p + d\varepsilon_z^p$$

$$\sqrt{J_2} \rightarrow d\gamma_0^p = \sqrt{\frac{1}{2}(de_x^{p2} + de_y^{p2} + de_z^{p2}) + \frac{1}{4}(d\gamma_{xy}^{p2} + d\gamma_{yz}^{p2} + d\gamma_{xz}^{p2})}$$

偏差ひずみ増分  $de_{ij}^p = d\varepsilon_{ij}^p - \frac{d\varepsilon_{kk}^p}{3}\delta_{ij}$

### 塑性ひずみ増分

$$d\gamma_0^p = |d\lambda \nabla f| \sin \theta$$
$$de_0^p = -|d\lambda \nabla f| \cos \theta$$



## 負荷曲面と関連流動則



# 複合硬化則

